Pair Production and Solutions of Wave Equation for Arbitrary-Spin Particles

S. I. Kruglov[1]

We investigate the theory of particles with arbitrary spin and magnetic moment in the Lorentz representation $(0, s) \oplus (s, 0)$ in an external constant and uniform electromagnetic field. We obtain the density matrix of free particles in pure spin states. The differential probability of pair producing particles with arbitrary spin by an external constant and uniform electromagnetic field is found using the exact solutions. We calculate the imaginary and real parts of the Lagrangian in an electromagnetic field that takes into account the vacuum polarization.

1. INTRODUCTION

Interest in the theory of relativistic particles with arbitrary spins is increasing. One of the reasons is that SUSY models require superpartners, i.e., additional fields of particles with higher spins. In particular, it is important to take into account particle with spin-3/2 gravitino which appears in cosmological models based on supergravity and in the theory of inflation of the universe [1].

Another factor is that some string models have similar features to models of relativistic spinning particles [2]. It is also interesting to have particles with arbitrary fractional spins [3] (see also refs. 4–7). Such spinning particles in $(2 + 1)$ dimensions, called anyons, were discovered and have anomalous statistics.

There are many different relativistic wave equations which describe particles with arbitrary spins [5–15]. The fields of free particles of a mass m and spin s in these formalisms realize definite representations of the Poincaré group. Some of these schemes are equivalent to each other. If the interaction with external electromagnetic fields is introduced, approaches

[1] International Educational Centre, Toronto, Ontario, Canada M3J 3G9.

based on different representations of the Lorentz group become inequivalent. Most theories of particles in external electromagnetic fields have difficulties such as noncausal propagation [16], indefinite metrics in the second-quantized theory [17–20], etc.

We proceed from the second-order relativistic wave equation for particles with arbitrary spin s and magnetic moment μ, on the basis of the Lorentz representation $(0, s) \oplus (s, 0)$, where $\oplus$ means the direct sum. In this scheme, the Hermitianizing matrix η exists for arbitrary spin, as there are pairs of mutually conjugate representations $(0, s)$ and $(s, 0)$ [13–15]. Therefore we have here a Lagrangian formulation and the scalar product of wave functions, which is important for any quantum mechanical calculations. Such an approach avoids difficulties of other schemes and the corresponding wave-function has the minimal number of components. This is a generalization of the Feynman–Gell-Mann equation [21] for particles with spin 1/2 to the case of higher spin particles which possess arbitrary magnetic moment. In the particular case of spin 1/2, we recover to the Dirac theory. If the normal magnetic moment is considered, it leads to the approach of refs. 13–15. Particles in this scheme propagate causally in external electromagnetic fields and this is a parity-symmetric theory with a Lagrangian formulation. Some equations on the basis of $(0, s) \oplus (s, 0)$ representations of the Lorentz group were studied in refs. 22–27. In refs. 13–15, the $2(6s + 1)$-component, first-order matrix parity-invariant formulation of the equation for particles with arbitrary spin was considered. Then the author obtained the second-order equation for particles with "normal" magnetic moment. Starting with the second-order equation for particles which possess arbitrary magnetic moment, we go to the first-order wave equation with another representation of the Lorentz group; the algebraic properties of the matrices are the same as in the approach of refs. 13–15. We find solutions of equations for free particles in the form of density matrices (projection matrix-dyads) for pure spin states which are used for different electromagnetic evaluation of the Feynman diagrams. Such projection matrix-dyads allow us to make covariant calculations without using matrices of the wave equation in a definite representation.

The main purpose of this paper is to investigate solutions of wave equations, pair production of arbitrary-spin particles by constant, uniform electromagnetic fields, and vacuum polarization of higher spin particles. Considering one-particle theory, and obtaining the differential probability for pair production of particles with arbitrary spins, we avoid the Klein paradox [28–30]. As a particular case of spin-1/2 and gyromagnetic ratio 2 particles, we arrive at the well-known result found by Schwinger [31], who predicted e^+e^- pair production in a strong external electromagnetic field. This has now been realized by the development of power laser techniques. It should be noted that the pair production of particles by a gravitational field is also

important for understanding the evolution of the universe near a singularity [32].

The probability of pair production of particles in external electromagnetic fields can be found on the basis of exact solutions of the wave equations [33–35] or the imaginary part of the Lagrangian [31]. We consider here both approaches. Nonlinear corrections to the Maxwell Lagrangian of the constant uniform electromagnetic fields are determined from the polarization of the vacuum of arbitrary-spin particles. The problem of pair production of particles with higher spins using the quasiclassical scheme (method of "imaginary time") was considered in ref. 36 and is in accord with our approach via relativistic wave equations. It should be noted that the quasiclassical approximation has a restriction for the fields E, $H \ll m^2/e$ when the process is exponentially suppressed. This means that the approach of ref. 36 is valid when the electromagnetic fields are not too strong, i.e., less than the critical value m^2/e. But it is known that pairs of particles are created rapidly at this critical value of the fields. In our consideration, there are no such restrictions. The problem of the pair production of vector particles with gyromagnetic ratio 2 was investigated in ref. 37. In refs. 38 and 39, the imaginary part of the effective Lagrangian which defines the probability of e^+e^- production was found by taking into account the anomalous magnetic moment.

We use the system of units $\hbar = c = 1$, $\alpha = e^2/4\pi = 1/137$, $e > 0$. In Section 2, proceeding from the second-order equation for arbitrary-spin particles with anomalous magnetic moment, we go to the first-order formulation of the theory. All independent solutions of the equation for free particles are found in the form of matrix-dyads (density matrices). In Section 3, we consider the important case of spin-1 particles, which is different from the Proca or Petiau–Duffin–Kemmer theories. Here, the $2(6s + 1)$-component first-order equation is constructed. Section 4 investigates exact solutions of arbitrary-spin particle equations in constant, uniform electromagnetic fields. On the basis of exact solutions, we find the differential probability of pair production of particles with arbitrary spin and anomalous magnetic moment. The imaginary part of the effective Lagrangian for electromagnetic fields is calculated. In Section 5, using the Schwinger method, we find the nonlinear corrections to the Lagrangian of a constant, uniform electromagnetic field caused by the vacuum polarization of particles with arbitrary spin and magnetic moment. Section 6 discusses the results.

2. WAVE EQUATION AND DENSITY MATRIX

We proceed here from the theory based on the $(s, 0) \oplus (0, s)$ Lorentz representation for massive particles. The wave function of the $(s, 0) \oplus (0, s)$

representation has $2(2s + 1)$ components. For spin 1/2, we arrive at well-known Dirac bispinors. For spin 1, however, there is doubling of the components compared with the Proca theory [40] because the vector particles have three spin states with the projections $s_z = \pm 1, 0$.

We postulate the following two (for $\varepsilon = \pm 1$) wave equations for arbitrary-spin particles in external electromagnetic fields:

$$\left(D_\mu^2 - m^2 - \frac{eq}{2s} F_{\mu\nu} \Sigma_{\mu\nu}^{(\varepsilon)} \right) \Psi_\varepsilon(x) = 0 \tag{1}$$

where s is the spin of particles, and $D_\mu = \partial_\mu - ie A_\mu$; $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the strength tensor of external electromagnetic fields, $\varepsilon = \pm 1$; and $\Sigma_{\mu\nu}^{(-)}$, $\Sigma_{\mu\nu}^{(+)}$ are the generators of the Lorenz group which correspond to the $(s, 0)$ and $(0, s)$ representations. The two equations in (1) (for $\varepsilon = \pm 1$) describe particles which possess the magnetic moment $\mu = eq/(2m)$ and gyromagnetic ratio $g = q/s$. At $q = 1$, we have the "normal" magnetic moment $\mu = e/(2m)$ and $g = 1/s$. The generators $\Sigma_{\mu\nu}^{(\varepsilon)}$ are connected with the spin matrices S_k by the relationships $\Sigma_{ij}^{(\varepsilon)} = \varepsilon_{ijk} S_k$ and $\Sigma_{4k}^{(\varepsilon)} = -i\varepsilon S_k$, where the parameter ε corresponds to the Lorentz group representations $(s, 0)$ for $\varepsilon = 1$ and $(0, s)$ for $\varepsilon = -1$. As usual, the relations

$$[S_i, S_j] = i\varepsilon_{ijk} S_k, \qquad (S_1)^2 + (S_2)^2 + (S_3)^2 = s(s + 1) \tag{2}$$

are valid, where $i, j, k = 1, 2, 3$; ε_{ijk} is the Levi-Civita symbol ($\varepsilon_{123} = 1$).

At $q = 1$, Eq. (1) was considered in refs. 13–15. The theory of arbitrary-spin particles based on Eq. (1) is causal in the presence of external electromagnetic fields. It is seen from the method of ref. 16 that the equations remain hyperbolic and the characteristic surfaces are lightlike. Equation (1) is invariant under the parity operation. Indeed, at the parity inversion $\varepsilon \rightarrow -\varepsilon$, the representation $(s, 0)$ transforms into $(0, s)$.

Now we will consider the problem of formulating a first-order relativistic wave equation from the second-order equation (1). This is convenient for some quantum electrodynamic calculations with polarized particles of arbitrary spins.

Let us introduce the matrix $\varepsilon^{A,B}$ with dimension $n \times n$; its elements consist of zeros and only one element is unity, where row A and column B cross. So the matrix elements and multiplication of this matrices are

$$(\varepsilon^{A,B})_{CD} = \delta_{AC} \delta_{BD}, \qquad \varepsilon^{A,B} \varepsilon^{C,D} = \delta_{BC} \varepsilon^{A,D} \tag{3}$$

where indexes $A, B, C, D = 1, 2, \ldots, n$.

Six generators $\Sigma_{\mu\nu}^{(+)}$ (or $\Sigma_{\mu\nu}^{(-)}$) occurring in Eq. (1) have the dimension $2s + 1$. Therefore the wave function $\Psi_+(x)$ [or $\Psi_-(x)$] of Eq. (1) possesses

$2s + 1$ components. Now we can introduce the $5(2s + 1)$-component wave function

$$\Psi_1(x) = \begin{pmatrix} \Psi_+(x) \\ -(1/m)D_\mu \, \Psi_+(x) \end{pmatrix} \tag{4}$$

so that $\Psi_1(x) = \{\Psi_A(x)\}$, $A = 0, \mu$; $\Psi_0 = \Psi_+(x)$, $\Psi_\mu = -(1/m)D_\mu \, \Psi_+(x)$ and $\Psi_+(x)$ realizes the Lorentz representation $(s, 0)$. It is not difficult to check that Eq. (1) for $\varepsilon = +1$ can be represented as the first-order equation

$$(\beta_\mu^{(+)} \, D_\mu + m)\Psi_1(x) = 0 \tag{5}$$

where $5(2s + 1) \times 5(2s + 1)$ matrices

$$\beta_\mu^{(+)} = (\varepsilon^{0,\mu} + \varepsilon^{\mu,0}) \otimes I_{2s+1} - i\frac{q}{s} \varepsilon^{0,\nu} \otimes \Sigma_{\mu\nu}^{(+)} \tag{6}$$

are introduced, and $\otimes$ is the direct product of matrices, I_{2s+1} is a unit matrix of the dimension $2s + 1$, and in (6), we imply the summation on index ν. It should be noted that in refs. 13–15, the $(6s + 1)$-component wave function was introduced for the case of "normal" magnetic moment of particles. Here a higher dimension $5(2s + 1)$ of the wave function is considered as compared with the $(6s + 1)$-component function investigated in refs. 13–15. The difference is that we do not introduce the vector potentials for arbitrary spin fields and deal only with the strength field tensor $\Psi_+(x)$ and its derivatives $D_\mu\Psi_+(x)$. The particular case of a spin-1 field will be considered in Section 3 on the basis of $(6s + 1)$-representation of the wave function by introducing vector potentials. Using properties (3), it is easy to check that the five-dimension matrices $\beta_\mu^{PDK} = \varepsilon^{0,\mu} + \varepsilon^{\mu,0}$ obey the Petiau–Duffin–Kemmer algebra [41–43],

$$\beta_\mu^{PDK} \beta_\nu^{PDK} \beta_\alpha^{PDK} + \beta_\alpha^{PDK} \beta_\nu^{PDK} \beta_\mu^{PDK} = \delta_{\mu\nu} \beta_\alpha^{PDK} + \delta_{\alpha\nu} \beta_\mu^{PDK} \tag{7}$$

The wave function $\Psi_1(x)$ transforms on the $[(0, 0) \oplus (1/2, 1/2)] \otimes (s, 0)$ representation of the Lorentz group [44, 45]. For the case $\varepsilon = -1$, we have the analogous equation

$$(\beta_\mu^{(-)}D_\mu + m)\Psi_2(x) = 0 \tag{8}$$

where

$$\beta_\mu^{(-)} = (\varepsilon^{\overline{0},\overline{\mu}} + \varepsilon^{\overline{\mu},\overline{0}}) \otimes I_{2s+1} - i\frac{q}{s} \varepsilon^{\overline{0},\overline{\nu}} \otimes \Sigma_{\mu\nu}^{(-)}$$

$$\Psi_2(x) = \begin{pmatrix} \Psi_-(x) \\ -(1/m)D_\mu \, \Psi_-(x) \end{pmatrix} \tag{9}$$

and $\Psi_2(x) = \{\Psi_B(x)\}$, $B = \bar{0}, \bar{\mu}$; $\Psi_{\bar{0}}(x) = \Psi_-(x)$, $\Psi_{\bar{\mu}}^- = -(1/m) D_\mu \Psi_-(x)$, where $\Psi_-(x)$ transforms as the $(0, s)$ representation of the Lorentz group, and $\Psi_2(x)$ realizes the $[(0, 0) \oplus (1/2, 1/2)] \otimes (0, s)$ representation. The two equations (5) and (8) can be combined into one first-order equation

$$(\beta_\mu D_\mu + m)\Psi(x) = 0 \tag{10}$$

with the matrices and wave function

$$\beta_\mu = \beta_\mu^{(+)} \oplus \beta_\mu^{(-)}, \qquad \Psi(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \end{pmatrix} \tag{11}$$

Using the properties of elements of the entire algebra (3), it is not difficult to verify that the matrices β_μ (the same for $\beta_\mu^{(+)}$ and $\beta_\mu^{(-)}$) obey the algebra

$$\beta_\mu\beta_\nu\beta_\sigma + \beta_\nu\beta_\sigma\beta_\mu + \beta_\sigma\beta_\mu\beta_\nu + \beta_\nu\beta_\mu\beta_\sigma + \beta_\mu\beta_\sigma\beta_\nu + \beta_\sigma\beta_\nu\beta_\mu$$
$$= 2(\delta_{\nu\sigma}\beta_\mu + \delta_{\mu\sigma}\beta_\nu + \delta_{\mu\nu}\beta_\sigma) \tag{12}$$

In other refs. 13–15, the $2(6s + 1)$-dimensional representation of the $SL(2, C)$ group was considered for particles of arbitrary spin with algebra (12). Here we study another representation of the $SL(2, C)$ group, and the matrices β_μ of (11) are $10(2s + 1) \times 10(2s + 1)$ dimensional. Different representations of this algebra were studied in refs. 46 and 47.

Let us consider the problem of finding the solutions to Eq. (10) for definite momentum and spin projection. It is convenient to find these solutions in the form of projection matrix-dyads (density matrices). All electrodynamic calculations of Feynman diagrams with arbitrary-spin particles can be done using these matrices. As particles in initial and final states are free particles, we can put the parameter $q = 0$ in (1), (10). This corresponds to the case when external electromagnetic fields are absent. Then the matrices β_μ transform to β_μ^0:

$$\beta_\mu^0 = [(\varepsilon^{0,\mu} + \varepsilon^{\mu,0}) \otimes I_{2s+1}] \oplus [(\varepsilon^{\bar{0},\bar{\mu}} + \varepsilon^{\bar{\mu},\bar{0}}) \otimes I_{2s+1}] \tag{13}$$

which obey the Petiau–Duffin–Kemmer algebra (7). The projection operators extracting states with definite 4-momentum p_μ for particle and antiparticle are given by

$$M_\pm = \frac{i\hat{p}(i\hat{p} \pm m)}{2m^2} \tag{14}$$

where $\hat{p} = p_\mu\beta_\mu^0$ (we use the metric such that $p^2 = \mathbf{p}^2 + p_4^2 = \mathbf{p}^2 - p_0^2 = -m^2$). Plus and minus signs in (14) correspond to the particle and antiparticle, respectively. Matrices $M_\pm$ have the usual projection operator property [48, 49]

$$M_\pm^2 = M_\pm \tag{15}$$

Equation (15) is verified by the relation $\hat{p}^3 = p^2\hat{p}$, which follows from the Petiau–Duffin–Kemmer algebra (7). To find the spin projection operators, we need the generators of the Lorentz group in the representation of the wave function $\Psi(x)$ which occurs in Eq. (10). From the structure of the functions $\Psi_1(x)$ and $\Psi_2(x)$ in (4) and (9), we define the generators of the Lorentz group in our $10(2s + 1)$-dimensional representation

$$J_{\mu\nu} = J_{\mu\nu}^{(+)} \oplus J_{\mu\nu}^{(-)}$$

$$J_{\mu\nu}^{(+)} = (\varepsilon^{\mu,\nu} - \varepsilon^{\nu,\mu}) \otimes I_{2s+1} + iI_5 \otimes \Sigma_{\mu\nu}^{(+)} \tag{16}$$

$$J_{\mu\nu}^{(-)} = (\varepsilon^{\overline{\mu},\overline{\nu}} - \varepsilon^{\overline{\nu},\overline{\mu}}) \otimes I_{2s+1} + iI_5 \otimes \Sigma_{\mu\nu}^{(-)}$$

where I_5 is the 5-dimensional unit matrix. Using properties (3), we get the commutation relations for $J_{\mu\nu}$, and

$$[J_{\mu\nu}, J_{\alpha\beta}] = \delta_{\nu\alpha}J_{\mu\beta} + \delta_{\mu\beta}J_{\nu\alpha} - \delta_{\mu\alpha}J_{\nu\beta} - \delta_{\nu\beta}J_{\mu\alpha} \tag{17}$$

$$[\beta_\mu, J_{\alpha\beta}] = \delta_{\mu\alpha}\beta_\beta - \delta_{\mu\beta}\beta_\alpha \tag{18}$$

Relationship (17) is a well-known commutation relation for generators of the Lorentz group [44, 45]. Equation (10) is a form invariant under the Lorentz transformations because relation (18) is valid. To guarantee the existence of a relativistically invariant bilinear form

$$\overline{\Psi}(x)\Psi(x) = \Psi^+(x)\eta\Psi(x) \tag{19}$$

where $\Psi^+(x)$ is the Hermitian-conjugate wave function, we should construct a Hermitianizing matrix η with the properties [46, 48, 49]:

$$\eta\beta_i = -\beta_i\eta, \qquad \eta\beta_4 = \beta_4\eta \qquad (i = 1, 2, 3) \tag{20}$$

Such a matrix exists because here there are two $(0, s)$ and $(s, 0)$ mutually conjugate representations; η is given by

$$\eta = (\varepsilon^{a,\overline{a}} + \varepsilon^{\overline{a},a} - \varepsilon^{4,\overline{4}} - \varepsilon^{\overline{4},4} - \varepsilon^{0,\overline{0}} - \varepsilon^{\overline{0},0}) \otimes I_{2s+1} \tag{21}$$

where the summation on the index $a = 1, 2, 3$ is implied. Now we introduce the operator of the spin projection on the direction of the momentum $\mathbf{p}$:

$$\hat{S}_\mathbf{p} = -\frac{i}{2|\mathbf{p}|} \varepsilon_{abc}p_aJ_{bc} = (\kappa_p + \sigma_p) \oplus (\overline{\kappa}_p + \overline{\sigma}_p) \tag{22}$$

where

$$\kappa_p = -\frac{i}{|\mathbf{p}|}\,\varepsilon_{abc}p_a\varepsilon^{b,c} \otimes I_{2s+1}, \qquad \overline{\kappa}_p = -\frac{i}{|\mathbf{p}|}\,\varepsilon_{abc}p_a\varepsilon^{\overline{b},\overline{c}} \otimes I_{2s+1}$$

$$\sigma_p = \overline{\sigma}_p = I_5 \otimes \frac{\mathbf{pS}}{|\mathbf{p}|} \tag{23}$$

and $|\mathbf{p}| = \sqrt{p_1^2 + p_2^2 + p_3^2}$. It is easy to check that the following commutation relation holds:

$$[\hat{S}_{\mathbf{p}}, \hat{p}] = 0$$

The matrices κ_p, $\overline{\kappa}_p$ obey the simple equations

$$\kappa_p^3 = \kappa_p, \qquad \overline{\kappa}_p^3 = \overline{\kappa}_p \tag{24}$$

Taking into account Eqs. (2), we derive relations for the matrices σ_p (23):

$$\left(\sigma_p^2 - \frac{1}{4}\right) \cdots (\sigma_p^2 - s^2) = 0 \qquad \text{for half-integer spins} \tag{25}$$

$$\sigma_p(\sigma_p^2 - 1) \cdots (\sigma_p^2 - s^2) = 0 \qquad \text{for integer spins}$$

Relations (25) allow us to construct projection operators which extract the pure spin states. Using the relationship

$$\hat{S}_{\mathbf{p}}\hat{p} = (\sigma_p \oplus \overline{\sigma}_p)\hat{p}$$

we can consider projection matrices on the basis of equations (25). The common technique of the construction of such operators is described in refs. 48 and 49. Let us consider the equation for the auxiliary spin operators σ_p, $\overline{\sigma}_p$ for the spin projection s_k:

$$\sigma_p \Psi_k = s_k \Psi_k \tag{26}$$

The solution to Eq. (26) can be found using relationships (25), which can be rewritten as

$$(\sigma_p - s_k)P_k(\sigma_p) = 0 \tag{27}$$

where the polynomials $P_k(\sigma_p)$ are given by

$$P_k(\sigma_p) = \left(\sigma_p^2 - \frac{1}{4}\right) \cdots (\sigma_p + s_k) \cdots (\sigma_p^2 - s^2) \qquad \text{for half-integer spins} \tag{28}$$

$$P_k(\sigma_p) = \sigma_p(\sigma_p^2 - 1) \cdots (\sigma_p + s_k) \cdots (\sigma_p^2 - s^2) \qquad \text{for integer spins}$$

Every column of the polynomial $P_k(\sigma_p)$ can be considered as an eigenvector Ψ_k of Eq. (26) with the eigenvalue s_k. As s_k is one multiple root of Eqs.

(25), all columns of the matrix $P_k(\sigma_p)$ are linearly independent solutions of Eq. (26) [48, 49]. It can be verified using definitions (28) that the matrix

$$Q_k = \frac{P_k(\sigma_p)}{P_k(s_k)} \tag{29}$$

is the projection operator with the relation

$$Q_k^2 = Q_k \tag{30}$$

Equation (30) tells that the matrix Q_k can be transformed into diagonal form, with the diagonal containing only ones and zeros. So the Q_k acting on the wave function Ψ will retain components which correspond to the spin projection s_k.

We have mentioned that this theory of arbitrary-spin particles involves the doubling of the spin states of particles because there are two representations $(s, 0)$ and $(0, s)$ of the Lorentz group. To separate these representations, which are connected by the parity transformations, we use the parity operator

$$K = (\varepsilon^{\mu,\bar{\mu}} + \varepsilon^{\bar{\mu},\mu} + \varepsilon^{0,\bar{0}} + \varepsilon^{\bar{0},0}) \otimes I_{2s+1} \tag{31}$$

with the summation on index $\mu = 1, 2, 3, 4$. The $10(2s + 1)$-dimensional matrix K has the property $K^2 = I_{10(2s+1)}$. The projection operator extracting states with the definite parity is given by

$$\Lambda_\varepsilon = \tfrac{1}{2}(K + \varepsilon) \tag{32}$$

where $\varepsilon = \pm 1$. This matrix possesses the required relationship

$$\Lambda_\varepsilon^2 = \Lambda_\epsilon \tag{33}$$

It should be noted that the matrix K in (31) plays a role analogous to the γ_5 matrix in the Dirac theory of particles with spin 1/2. It is easily verified that the operators $\hat{p}$, $\hat{S}_{\mathbf{p}}$, K commute with each other, and, as a consequence, they have common eigenvectors. The projection operator extracting pure states with definite 4-momentum projection, spin, and parity is given by

$$\Pi_{\pm m,k,\varepsilon} = M_\pm \Lambda_\varepsilon (Q_k \oplus Q_k) \tag{34}$$

with matrices (14), (29), and (32). The $\Pi_{\pm m,k,\varepsilon}$ is the density matrix for pure states. It is easy to consider impure (mixed) states by summation of (34) over definite quantum numbers s_k, ε. The projection operator for pure states can be represented as matrix-dyad [48, 49]

$$\Pi_{\pm m,k,\varepsilon} = \Psi^\varepsilon_{\pm m,k} \cdot \overline{\Psi}^\varepsilon_{\pm m,k} \tag{35}$$

where $\overline{\Psi}^\varepsilon_{\pm m,k} = (\Psi^\varepsilon_{\pm m,k})^+ \eta$, and $\Psi^\varepsilon_{\pm m,k}$ is the solution to the equations

$$(i\beta_\mu p_\mu \pm m)\Psi^\epsilon_{\pm m,k} = 0, \qquad \hat{S}_\mathbf{p}\Psi^\epsilon_{\pm m,k} = s_k \Psi^\epsilon_{\pm m,k}$$

$$K\Psi^\epsilon_{\pm m,k} = \epsilon \Psi^\epsilon_{\pm m,k} \tag{36}$$

Expression (35) is convenient for calculations for different quantum electrodynamics processes with polarized particles of arbitrary spins.

3. WAVE EQUATION FOR SPIN-1 PARTICLES

As a particular case, we consider the wave equation for particles with spin 1. To compare our approach with refs. 13–15, the $2(6s + 1)$ representation for the wave function will be studied. Equation (1) in the case of spin-1 particles becomes

$$(D_\lambda^2 - m^2)\Psi_{\mu\nu}(x) - ieq\delta_{[\mu\nu][\alpha\beta]}F_{\alpha\sigma}\Psi_{\sigma\beta}(x) = 0 \tag{37}$$

where the wave function $\Psi_{\mu\nu}(x)$ is an antisymmetric tensor of the second rank $\Psi_{\mu\nu}(x) = -\Psi_{\nu\mu}(x)$, which has six independent components, and $\delta_{[\mu\nu][\alpha\beta]} = \delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}$. For comparison with refs. 13–15, the "normal" magnetic moment $\mu = e/2m$ will be considered here. Then the parameter $q = 1$, and gyromagnetic ratio $g = 1$. Let us introduce two 4-vector potentials $\psi_\mu(x)$ and $\tilde{\psi}_\mu(x)$ in accordance with the relationship [50]

$$\Psi_{\mu\nu}(x) = D_\mu\psi_\nu(x) - D_\nu\psi_\mu(x) - \epsilon_{\mu\nu\alpha\beta}D_\alpha\tilde{\psi}_\beta(x) \tag{38}$$

where $\epsilon_{\mu\nu\alpha\beta}$ is an antisymmetric Levi-Civita tensor, $\epsilon_{1234} = -i$. This is a more general representation of the antisymmetric tensor $\Psi_{\mu\nu}(x)$ via 4-vector $\psi_\mu(x)$ and 4-pseudovector $\tilde{\psi}_\mu(x)$. It is not difficult to verify that Eq. (37) (at $q = 1$) is a consequence of the equations

$$D_\nu\Psi_{\mu\nu}(x) + m^2\psi_\mu(x) = 0 \tag{39}$$

$$D_\nu\tilde{\Psi}_{\mu\nu}(x) + m^2\tilde{\psi}_\mu(x) = 0 \tag{40}$$

where $\tilde{\Psi}_{\mu\nu}(x) = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\Psi_{\mu\nu}(x)$ is the dual tensor. So, the system of the first-order equations (38)–(40) is equivalent to the second-order equation (37). A doubling of spin states of particles is obvious here. It follows from Eqs. (39) and (40) for free particles (when external electromagnetic fields are absent and $D_\mu = \partial_\mu - ieA_\mu \to \partial_\mu$) that the Lorentz equations $\partial_\mu\psi_\mu(x) = 0$, $\partial_\mu\tilde{\psi}_\mu(x) = 0$ hold. Therefore the vector potential $\psi_\mu(x)$ describes spin-1 states (the state with spin 0 is not present due to the Lorentz equation) and pseudovector $\tilde{\psi}_\mu(x)$ also describes spin-1 states. As a result, there are six spin states and the system of equations (38)–(40) is not equivalent to the Proca theory [40]. If we set $\tilde{\psi}_\mu(x) = 0$ in (38)–(40), we arrive at the Proca equations.

Equations (38)–(40) are equivalent to the following system:

$$D_\nu M_{\mu\nu} + m^2 M_\mu = 0$$

$$M_{\mu\nu} = D_\mu M_\nu - D_\nu M_\mu - \epsilon_{\mu\nu\alpha\beta} D_\alpha M_\beta \tag{41}$$

with the self-dual tensor $M_{\mu\nu} = i\tilde{M}_{\mu\nu}$ and

$$D_\nu N_{\mu\nu} + m^2 N_\mu = 0$$

$$N_{\mu\nu} = D_\mu N_\nu - D_\nu N_\mu + \epsilon_{\mu\nu\alpha\beta} D_\alpha N_\beta \tag{42}$$

with the anti-self-dual tensor $N_{\mu\nu} = -i\tilde{N}_{\mu\nu}$, where

$$M_\mu = \tfrac{1}{2}(\psi_\mu(x) - i\tilde{\psi}_\mu(x)), \qquad M_{\mu\nu} = \tfrac{1}{2}(\Psi_{\mu\nu}(x) + i\tilde{\Psi}_{\mu\nu}(x))$$

$$N_\mu = \tfrac{1}{2}(\psi_\mu(x) + i\tilde{\psi}_\mu(x)), \qquad N_{\mu\nu} = \tfrac{1}{2}(\Psi_{\mu\nu}(x) - i\tilde{\Psi}_{\mu\nu}(x))$$

Adding and subtracting Eqs. (41), (42), we get Eqs. (38)–(40). The self-dual tensor $M_{\mu\nu}$ which obeys Eqs. (41) is transformed under (1, 0) representation of the Lorentz group and has three independent components. Equations (41) are not invariant to the parity transformation and there is no Lagrangian formulation of them. This also applies to Eqs. (42) for the anti-self-dual tensor $N_{\mu\nu}$, which transforms under (0, 1) representation of the Lorentz group. But if we consider the whole system of equations (41), (42) [which is equivalent to Eqs. (38)–(40)] on the basis of the (1, 0) $\oplus$ (0, 1) representation of the Lorentz group, we will have P-invariant theory within the Lagrangian formulation.

Consider now the first-order formulation of Eqs. (38)–(40) in matrix form (10). Introducing a 14-dimensional wave function

$$\Psi(x) = \{\Psi_A(x)\} = \begin{pmatrix} \psi_\mu(x) \\ \Psi_{\mu\nu}(x) \\ \tilde{\psi}_\mu(x) \end{pmatrix} \tag{43}$$

where index $A = \mu$, $[\mu\nu]$, $\tilde{\mu}$, so that $\Psi_\mu(x) \equiv \psi_\mu(x)$, $\Psi_{[\mu\nu]}(x) \equiv \Psi_{\mu\nu}(x)$, $\Psi_{\tilde{\mu}}(x) \equiv \tilde{\psi}_\mu(x)$, we have that Eqs. (38)–(40) take the form (10) with the matrices

$$\beta_\mu = \beta_\mu^{(1)} + \beta_\mu^{(2)} \tag{44}$$

$$\beta_\mu^{(1)} = \varepsilon^{\lambda,[\lambda\mu]} + \varepsilon^{[\lambda\mu],\lambda} \tag{45}$$

$$\beta_\mu^{(2)} = \tfrac{1}{2}\varepsilon_{\lambda\mu\alpha\beta}(\varepsilon^{\tilde{\lambda},[\alpha\beta]} + \varepsilon^{[\alpha\beta],\tilde{\lambda}}) \tag{46}$$

where $[\mu\nu]$ means the antisymmetric combination of indexes μ and ν and corresponds to the six-dimensional subspace and there is a summation on the repeating indexes λ in (45), (46). Here the representation with the dimension $2(6s + 1) = 14$ is valid. The $(6s + 1)$ representation is built from M_μ and the self-dual tensor $M_{\mu\nu}$, or from N_μ and the anti-self-dual tensor $N_{\mu\nu}$. The

resulting $2(6s + 1)$ representation (43) is the direct sum of the above representations. The fact that the matrices (44) obey the algebra (12) is confirmed by the equalities (3). This representation is different from the $10(2s + 1)$-dimensional representation (11). For spin-1 particles, the wave function (11) has $10(2s + 1) = 30$ components and is given by

$$\Psi(x) = \begin{pmatrix} \Psi_{\alpha\beta}(x) \\ -\dfrac{1}{m} D_\mu \Psi_{\alpha\beta}(x) \end{pmatrix}$$

The approach considered in Section 2 allows us to apply the technique for arbitrary-spin particles which possess arbitrary magnetic moment without introducing "potentials", but the cost of this is the high dimension $10(2s + 1)$.

It should be noted that 10-dimensional matrices $\beta_\mu^{(1)}$ in (45) obey the Petiau–Duffin–Kemmer algebra (7), and that they act in the 10-dimensional subspace of wave functions

$$\Psi^{\text{PDK}}(x) = \begin{pmatrix} \psi_\mu(x) \\ \Psi_{\mu\nu}(x) \end{pmatrix}$$

where we imply that $\Psi_{\mu\nu}(x) = D_\mu \psi_\nu(x) - D_\nu \psi_\mu(x)$, so that the 10-dimensional Petiau–Duffin–Kemmer equation is given by

$$(\beta_\mu^{(1)} D_\mu + m)\, \Psi^{\text{PDK}}(x) = 0 \tag{47}$$

This corresponds to spin-1 particles without the doubling of spin states.

The generators of the Lorentz group in our 14-dimensional representation (43) are given by

$$J_{\mu\nu} = \varepsilon^{\mu,\nu} - \varepsilon^{\nu,\mu} + \varepsilon^{[\lambda\mu],[\lambda\nu]} - \varepsilon^{[\lambda\nu],[\lambda\mu]} + \varepsilon^{\bar\mu,\bar\nu} - \varepsilon^{\bar\nu,\bar\mu} \tag{48}$$

They obey the required commutation relations (17), (18). The projection operators extracting states with definite 4-momentum p_μ have virtually the same form (14), but with matrices (44). The operator for the spin projection

$$\hat{S}_{\mathbf{p}} = -\frac{i}{2|\mathbf{p}|}\, \epsilon_{abc}\, p_a\, J_{bc} \tag{49}$$

with generators (48) satisfies the equation

$$\hat{S}_{\mathbf{p}}(\hat{S}_{\mathbf{p}}^2 - 1) = 0 \tag{50}$$

In accordance with this procedure [see (27)–(30)], we find the projection operators corresponding to the definite spin projections

$$\hat{Q}_\pm = \tfrac{1}{2}\hat{S}_{\mathbf{p}}(\hat{S}_{\mathbf{p}} \pm 1) \tag{51}$$

$$\hat{Q}_0 = 1 - \hat{S}_{\mathbf{p}}^2 \tag{52}$$

Operators $\hat{Q}_\pm$, $\hat{Q}_0$ extract the spin projections $s_k = \pm 1$ and $s_k = 0$, respectively. Equations (38)–(40) are invariant under the dual $SO(2)$ transformations

$$\psi'_\mu(x) = \psi_\mu(x) \cos \alpha + \tilde{\psi}_\mu(x) \sin \alpha$$

$$\tilde{\psi}'_\mu(x) = -\psi_\mu(x) \sin \alpha + \tilde{\psi}_\mu(x) \cos \alpha \tag{53}$$

This group of symmetry is connected with the doubling of spin states of particles. To remove this degeneracy, we use the projection generator (32) with the matrix

$$K = \varepsilon^{\mu,\bar{\mu}} + \varepsilon^{\bar{\mu},\mu} + \tfrac{1}{4}\epsilon_{\mu\nu\alpha\beta}\,\varepsilon^{[\mu\nu],[\alpha\beta]} \tag{54}$$

which has the property $K^2 = I_{14}$, where I_{14} is the unit matrix in the 14-dimensional space. Now we construct the density matrices of spin-1 particles for pure states as products of the projection operators:

$$\Pi_{\pm m,k,\varepsilon} = \frac{1}{8m^2}\,\hat{S}_{\mathbf{p}}(\hat{S}_{\mathbf{p}} + s_k)i\hat{p}(i\hat{p} \pm m)(K + \varepsilon) \tag{55}$$

$$\Pi_{\pm m,0,\varepsilon} = \frac{1}{4m^2}\,(1 - \hat{S}_{\mathbf{p}}^2)i\hat{p}(i\hat{p} \pm m)(K + \varepsilon) \tag{56}$$

where $\hat{p} = p_\mu\beta_\mu$, and the spin projections on the direction of the momentum $\mathbf{p}$ are $s_k = \pm 1$ in (55) and $s_k = 0$ in (56), and $\varepsilon = \pm 1$. Matrices (55), (56) can be represented in the form of matrix-dyads (35) with the Hermitianizing matrix

$$\eta = \varepsilon^{m,m} - \varepsilon^{4,4} + \varepsilon^{[m4],[m4]} - \tfrac{1}{2}\varepsilon^{[mn],[mn]} - \varepsilon^{\bar{m},\bar{m}} + \varepsilon^{\bar{4},\bar{4}} \tag{57}$$

where $m, n = 1, 2, 3$, which obeys the general equations (20) with matrices (44). The Lagrangian of free particles (where the vector potential of the electromagnetic field $A_\mu = 0$) takes the standard form of

$$\mathscr{L} = -\,\overline{\Psi}(x)(\beta_\mu\partial_\mu + m)\Psi(x)$$

where $\overline{\Psi}(x) = \Psi^+(x)\eta$, and $\Psi^+(x)$ is the Hermitian-conjugate wave function.

4. PAIR PRODUCTION BY EXTERNAL ELECTROMAGNETIC FIELDS

To calculate the probability of pair production of arbitrary-spin particles, we follow the Nikishov method [33–35]. Thus, exact solutions to Eq. (1) should be found for external, constant, uniform electromagnetic fields. Using the properties of generators $\Sigma_{\mu\nu}^{(\epsilon)}$, we find the relationships

$$\tfrac{1}{2}\Sigma_{\mu\nu}^{(+)} F_{\mu\nu} = S_i X_i, \qquad \tfrac{1}{2}\Sigma_{\mu\nu}^{(-)} F_{\mu\nu} = S_i X_i^* \tag{58}$$

where $X_i = H_i + iE_i$ and $X_i^* = H_i - iE_i$; E_i, H_i are the electric and magnetic

fields, respectively, and the spin matrices S_i obey Eqs. (2). In the diagonal representation, the equations for the eigenvalues are

$$S_i X_i \Psi_+^{(\sigma)}(x) = \sigma X \Psi_+^{(\sigma)}(x), \qquad S_i X_i^* \Psi_-^{(\sigma)}(x) = \sigma X^* \Psi_-^{(\sigma)}(x) \qquad (59)$$

where $X = \sqrt{\mathbf{X}^2}$, $\mathbf{X} = \mathbf{H} + i\mathbf{E}$, and the spin projection σ is

$$\sigma = \begin{cases} \pm s, \pm(s-1), \ldots, 0 & \text{for integer spins} \\ \pm s, \pm(s-1), \ldots, \pm\frac{1}{2} & \text{for half-integer spins} \end{cases} \qquad (60)$$

Taking into account (58) and (59), we rewrite Eq. (1) (for $\varepsilon = \pm 1$) as

$$(D_\mu^2 - m^2 - a\sigma X)\,\Psi_+^{(\sigma)}(x) = 0, \qquad (D_\mu^2 - m^2 - a\sigma X^*)\Psi_-^{(\sigma)}(x) = 0 \qquad (61)$$

where $^\alpha = eq/s$. These equations are like the Klein–Gordon equation for scalar particle, but with complex "effective" masses $m_{\text{eff}}^2 = m^2 + a\sigma X$ and $(m_{\text{eff}}^2)^* = m^2 + a\sigma X^*$. It is sufficient to consider only one of Eqs. (61). Let us consider the solution of the equation

$$(D_\mu^2 - m_{\text{eff}}^2)\Psi^{(\sigma)}(x) = 0 \qquad [\Psi^{(\sigma)}(x) \equiv \Psi_+^{(\sigma)}(x)] \qquad (62)$$

in the presence of the external, constant, uniform electromagnetic fields. The general case is when there are two Lorentz invariants of the electromagnetic fields $\mathcal{F} = \frac{1}{4}F_{\mu\nu}^2 \neq 0$ and $\mathcal{G} = \frac{1}{4}F_{\mu\nu}\tilde{F}_{\mu\nu} \neq 0$ ($\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$; $\epsilon_{\mu\nu\alpha\beta}$ is the antisymmetric Levi-Civita tensor). Then there is a coordinate system in which the electric $\mathbf{E}$ and magnetic $\mathbf{H}$ fields are parallel, i.e., $\mathbf{E} \parallel \mathbf{H}$. In this case, the 4-vector potential is given by

$$A_\mu = (0, x_1 H, -x_0 E, 0) \qquad (63)$$

so that 3-vectors $\mathbf{E} = \mathbf{n}E$ and $\mathbf{H} = \mathbf{n}H$ are directed along the axes, where $\mathbf{n} = (0, 0, 1)$ is a unit vector. The four solutions of Eq. (62) for the potential (63) with different asymptotic forms are given in refs. 33–35 and 51 (see also refs. 52 and 53)

$$_\pm^\pm\Psi_{p,n}^{(\sigma)}(x) = N \exp\left\{i(p_2 x_2 + p_3 x_3) - \frac{\eta^2}{2}\right\}H_n(\eta)\,_\pm^\pm\psi^{(\sigma)}(\tau) \qquad (64)$$

where N is the normalization constant, $H_n(\eta)$ is the Hermite polynomial,

$$\eta = \frac{eHx_1 + p_2}{\sqrt{eH}}, \qquad \nu = \frac{ik^2}{2eE} - \frac{1}{2}, \qquad \tau = \sqrt{eE}\left(x_0 + \frac{p_0}{eE}\right)$$

and

$$_+\psi^{(\sigma)}(\tau) = D_\nu[-(1-i)\tau], \qquad {}^-\psi^{(\sigma)}(\tau) = D_\nu[(1-i)\tau]$$

$$^+\psi^{(\sigma)}(\tau) = D_{\nu*}[(1+i)\tau], \qquad _-\psi^{(\sigma)}(\tau) = D_{\nu*}[-(1+i)\tau] \qquad (65)$$

Here $D_\nu(x)$ is the Weber–Hermite function (the parabolic-cylinder function). The probability for pair production of particles with arbitrary spins by constant electromagnetic fields can be obtained through the asymptotic form of the solutions (65) when the time $x_0 \to \pm\infty$. At $x_0 \to \pm\infty$, the functions $^{+}_{+}\psi^{(\sigma)}(\tau)$ have positive frequency and $^{-}_{+}\psi^{(\sigma)}(\tau)$ have negative frequency. The constant k^2 which enters the index ν of the parabolic-cylinder functions (65) is given by [52, 53]

$$k^2 = m_{\text{eff}}^2 + eH(2n + 1) \qquad (66)$$

where $n = l + r$, l is the orbital quantum number, r is the radial quantum number, and $n = 0, 1, 2, \ldots$ is the principal quantum number. It should be noted that for scalar particles, we have the equation $k^2 = p_0^2 - p_3^2$, where p_0 is the energy and p_3 is the third projection of the momentum of a scalar particle. In our case of arbitrary-spin particles, the parameter m_{eff}^2 is a complex value. Nevertheless all physical quantities in this case are real values. Solutions (64), (65) are characterized by three conserved numbers: k^2 and the momentum projections p_2, p_3. As shown in refs. 33–35, the functions (64) are connected by the relations

$$_{+}\Psi_{p,n}^{(\sigma)}(x) = c_{1n\sigma}\, ^{+}\Psi_{p,n}^{(\sigma)}(x) + c_{2n\sigma}\, ^{-}\Psi_{p,n}^{(\sigma)}(x)$$

$$^{+}\Psi_{p,n}^{(\sigma)}(x) = c_{1n\sigma}^{*}\, _{+}\Psi_{p,n}^{(\sigma)}(x) - c_{2n\sigma}\, _{-}\Psi_{p,n}^{(\sigma)}(x)$$

$$^{-}\Psi_{p,n}^{(\sigma)}(x) = -c_{2n\sigma}^{*}\, _{+}\Psi_{p,n}^{(\sigma)}(x) + c_{1n\sigma}\, _{-}\Psi_{p,n}^{(\sigma)}(x) \qquad (67)$$

$$_{-}\Psi_{p,n}^{(\sigma)}(x) = c_{2n\sigma}^{*}\, ^{+}\Psi_{p,n}^{(\sigma)}(x) + c_{1n\sigma}^{*}\, ^{-}\Psi_{p,n}^{(\sigma)}(x)$$

where coefficients $c_{1n\sigma}$, $c_{2n\sigma}$ are given by

$$c_{2n\sigma} = \exp\left[-\frac{\pi}{2}(\lambda + i)\right], \qquad \lambda = \frac{m_{\text{eff}}^2 + eH(2n + 1)}{eE}$$

$$|c_{1n\sigma}|^2 - |c_{2n\sigma}|^2 = 1 \qquad \text{for bosons} \qquad (68)$$

$$|c_{1n\sigma}|^2 + |c_{2n\sigma}|^2 = 1 \qquad \text{for fermions}$$

The values $c_{1n\sigma}$, $c_{2n\sigma}$ are connected with the probability of pair production of arbitrary-spin particles in the state with the quantum number n and the spin projection σ. The absolute probability for the production of a pair in the state with quantum number n, momentum p, and the spin projection σ throughout all space and during all time is

$$|c_{2n\sigma}|^2 = \exp\left\{-\pi\left[\frac{m^2}{eE} + \frac{q\sigma H}{sE} + \frac{H}{E}(2n + 1)\right]\right\} \qquad (69)$$

The value (69) is also the probability of the annihilation of a pair with

quantum numbers n, p, σ with the energy transfer to the external electromagnetic field. It is seen from (69) that for $H \gg E$, the pair of particles are mainly created by the external fields in the state with $n = 0$, $\sigma = -s$. This is the state with the smallest energy. So at $H \gg E$ there is a production of polarized beams of particles and antiparticles with the spin projection $\sigma = -s$ (s is the spin of particles). The average number of pairs of particles produced from a vacuum is

$$\overline{N} = \int \sum_{n,\sigma} |c_{2n\sigma}|^2 \, dp_2 \, dp_3 \, \frac{L^2}{(2\pi)^2} \tag{70}$$

because $(2\pi)^{-2} \, dp_2 \, dp_3 L^2$ is the density of final states, where L is the cutoff along the coordinates, i.e., L^3 is the normalization volume. The variables η, τ define the region where the process occurs, which is described by solutions (64) with the coordinates of the center of this region $x_0 = -p_3/eE$, $x_1 = -p_2/eH$. Therefore instead of the integration over p_2 and p_3 in (70), it is possible to use the substitutions [33–35]

$$\int dp_2 \to eHL, \qquad \int dp_3 \to eET \tag{71}$$

with T being the time of observation.

Evaluating the sum in (70) over the principal quantum number n, with the help of (69), (71), we obtain the probability of pair production per unit volume and per unit time

$$I(E, H) = \frac{\overline{N}}{VT} = \frac{e^2 EH}{8\pi^2} \frac{\exp[-\pi m^2/(eE)]}{\sinh(\pi H/E)} \sum_{\sigma} \exp(-\sigma b) \tag{72}$$

where $b = \pi qH/(sE)$, $V = L^3$. Now we evaluate the sum over the spin projection σ in (72) for integer and half-integer spins:

1. Integer spins:

$$\sum_{\sigma} \exp(-\sigma b) = S_1 + S_2, \qquad S_1 = e^0 + e^{-b} + \cdots + e^{-sb} = \frac{e^{-b(s+1)} - 1}{e^{-b} - 1}$$

$$S_2 = e^b + e^{2b} + \cdots + e^{sb} = \frac{e^b(e^{bs} - 1)}{e^b - 1} \tag{73}$$

$$S_1 + S_2 = \frac{\cosh(sb) - \cosh[(s+1)b]}{1 - \cosh b}$$

2. Half-integer spins:

$$\sum_{\sigma} \exp(-\sigma b) = S_1' + S_2',$$

$$S_1' = e^{-b/2} + e^{-3b/2} + \cdots + e^{-sb} = \frac{e^{-b(s+1)} - e^{-b/2}}{e^{-b} - 1} \tag{74}$$

$$S_2' = e^{b/2} + e^{3b/2} + \cdots + e^{sb} = \frac{e^{b(s+1)} - e^{b/2}}{e^{b} - 1}$$

$$S_1' + S_2' = \frac{\cosh(sb) - \cosh[(s+1)b]}{1 - \cosh b}$$

So the final expressions for integer and half-integer spins, (73) and (74), are the same. Using the relationship

$$\frac{\cosh(sb) - \cosh[(s+1)b]}{1 - \cosh b} = \cosh(sb) + \sinh(sb)\coth\frac{b}{2}$$

$$= \frac{\sinh[b(s+1/2)]}{\sinh(b/2)} \tag{75}$$

and Eqs. (72), (73) we arrive at the pair-production probability

$$I(E, H) = \frac{\overline{N}}{VT} = \frac{e^2\,EH}{8\pi^2} \frac{\exp[-\pi m^2/(eE)]}{\sinh(\pi H/E)} \frac{\sinh[(2s+1)q\pi H/(2sE)]}{\sinh[q\pi H/(2sE)]} \tag{76}$$

Expression (76) coincides with that derived in refs. 36 using the quasi-classical approach. So $I(E, H)$ is the intensity of the creation of pairs of arbitrary-spin particles which possess the magnetic moment $\mu = eq/(2m)$ and gyromagnetic ratio $g = q/s$.

In ref. 36, there is a discussion of physical consequences which follow from Eq. (76). In particular, there is a pair production in a purely magnetic field if $q = gs > 1$ [36]. It is interesting that the exact formula derived here from quantum field theory (which is valid for arbitrary fields E, H) coincides with the asymptotic expression obtained in ref. 36 for $E, H \ll m^2/e$.

To obtain the imaginary part of the density of the Lagrangian, we use the relationship [33–35]

$$VT\,\mathrm{Im}\,\mathscr{L} = \frac{1}{2}\int \sum_{n,\sigma} \ln|c_{1n\sigma}|^2\,dp_2\,dp_3\,\frac{L^2}{(2\pi)^2} \tag{77}$$

With the help of (68), (71), we arrive at (see also ref. 36)

$$\text{Im } \mathscr{L} = \frac{e^2 \, EH}{16\pi^2} \sum_{n=1}^{\infty} \frac{\beta_n}{n} \exp\left(-\frac{\pi m^2 n}{eE}\right) \frac{\sinh[n(2s+1)q\pi H/(2sE)]}{\sinh(n\pi H/E) \, \sinh[nq\pi H/(2sE)]}$$

$$(78)$$

where

$$\beta_n = \begin{cases} (-1)^{n-1} & \text{for bosons} \\ 1 & \text{for fermions} \end{cases}$$

The different expressions for bosons and fermions occur due to different statistics and relations (68). The first term ($n = 1$) in (78) coincides with the probability of the pair production per unit volume per unit time divided by 2 [33–35] (see discussion in ref. 36).

5. VACUUM POLARIZATION OF ARBITRARY-SPIN PARTICLES

Now we calculate the nonlinear corrections to the Lagrangian of a constant, uniform electromagnetic field interacting with a vacuum of arbitrary-spin particles with gyromagnetic ratio g. For the case of spins 0, 1/2, and 1 (for $g = 2$) this problem was solved in refs. 31, 37, 54, and 55. The nonlinear corrections to the Lagrangian of the electromagnetic field describe the effect of scattering of light by light. We consider one-loop corrections to the Maxwell Lagrangian corresponding to arbitrary-spin particles, and, to take into account vacuum polarization, it is convenient to adapt the Schwinger method [31]. Applying this approach to the arbitrary spin particles described by Eq. (1), we arrive at the effective Lagrangian of constant, uniform electromagnetic fields,

$$\mathscr{L}_1 = -\frac{1}{32\pi^2} \int_0^{\infty} d\tau \, \tau^{-3} \exp(-m^2\tau - l(\tau)) \, tr \, \exp\left(\frac{eq}{2s} \Sigma_{\mu\nu} F_{\mu\nu}\tau\right) \quad (79)$$

where

$$\Sigma_{\mu\nu} = \Sigma_{\mu\nu}^{(+)} \oplus \Sigma_{\mu\nu}^{(-)}, \qquad l(\tau) = \tfrac{1}{2} \, tr \, \ln[(eF\tau)^{-1} \sin(eF\tau)] \qquad (80)$$

and $F_{\mu\nu}$ is a constant tensor of the electromagnetic field. The formal difference of (79) from the case of spin-1/2 particles is in the substitution $\sigma_{\mu\nu} \rightarrow (q/s)\Sigma_{\mu\nu}$, where $\sigma_{\mu\nu} = (i/2)[\gamma_\mu, \gamma_\nu]$, γ_μ being the Dirac matrices. The problem is to calculate the trace of the matrices occurring in the exponential factor in (79). Using relations (58)–(60) and (73)–(75), we find

$$tr \, \exp\!\left(\frac{eq}{2s} \, \Sigma_{\mu\nu} \, F_{\mu\nu}\tau\right)$$

$$= 2 \, \text{Re} \left[\cosh(eqX\tau) + \sinh(eqX\tau) \, \coth\!\left(\frac{eqX\tau}{2s}\right) \right] \qquad (81)$$

Inserting (81) into (79) and adding the constant, which is necessary to cancel $\mathcal{L}_1$ when the electromagnetic fields are turned off [31], we arrive at

$$\mathcal{L}_1 = -\frac{1}{8\pi^2} \int_0^\infty d\tau \, \tau^{-3} \, \exp(-m^2\tau)$$

$$\times \left[(e\tau)^2 \mathcal{G} \, \frac{\text{Re}[\cosh(eqX\tau) + \sinh(eqX\tau) \coth(eqX\tau/2s)]}{2 \, \text{Im} \cosh(eX\tau)} - \frac{2s+1}{2} \right] \qquad (82)$$

where $\mathcal{G} = \mathbf{EH}$. With $q = 1$ and $s = 1/2$, this Lagrangian (82) coincides with that of Schwinger[31]. Expression (82) is the correction to the Maxwell Lagrangian that takes into account the vacuum polarization of arbitrary-spin particles which possess the magnetic moment $\mu = eq/(2m)$ and gyromagnetic ratio $g = q/s$. Adding (82) to the Lagrangian of the free electromagnetic fields

$$\mathcal{L}_0 = -\mathcal{F} = \frac{1}{2} \, (\mathbf{E}^2 - \mathbf{H}^2)$$

and introducing the divergent constant for weak fields, we get the expression for the total Maxwell Lagrangian

$$\mathcal{L}_M = \mathcal{L}_0 + \mathcal{L}_1 = -Z\mathcal{F} - \frac{1}{8\pi^2} \int_0^\infty d\tau \, \tau^{-3} \, \exp(-m^2\tau)$$

$$\times \left[(e\tau)^2 \mathcal{G} \, \frac{\text{Re}[\cosh(eqX\tau) + \sinh(eqX\tau) \coth(eqX\tau/2s)]}{2 \, \text{Im} \cosh(eX\tau)} \right.$$

$$\left. - \frac{2s+1}{2} - 4\beta(e\tau)^2\mathcal{F} \right] \qquad (83)$$

where

$$Z = 1 + \frac{e^2\beta}{2\pi^2} \int_0^\infty d\tau \, \tau^{-1} \, \exp(-m^2\tau), \qquad \beta = \frac{(2s+1) \, [s(q^2-1) + q^2]}{24s}$$

$$(84)$$

Schwinger's procedure is used to renormalize the electromagnetic field

$\mathscr{F} \to Z\mathscr{F}$ and the charge $e \to Z^{-1/2}e$. After expanding (83) in the small electric E and magnetic H fields, we arrive at the Lagrangian of a constant, uniform electromagnetic field (in rational units)

$$\mathscr{L}_{\mathrm{M}} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{H}^2)$$

$$+ \frac{2\alpha^2}{45m^4}\left[(\mathbf{E}^2 - \mathbf{H}^2)^2(15\beta - \gamma) + (\mathbf{EH})^2\left(4\gamma + \frac{2s + 1}{2}\right)\right]$$

$$+ \ldots \tag{85}$$

where $\alpha = e^2/(4\pi)$ and

$$\gamma = \frac{(2s + 1)\,[q^4(s + 1)\,(3s^2 + 3s - 1) - 3s^3]}{16s^3} \tag{86}$$

It is easy to verify that for the particular case of $s = 1/2$, $q = 1$ (which corresponds to the Dirac theory), Eq. (85) coincides with the well-known Schwinger Lagrangian [31]. With $s = 1$ and $q = 2$, expression (85) is different from one obtained in ref. 37. This is because our theory of particles with $s = 1$ is not equivalent to the Proca theory (see Section 3); there is the doubling of spin states here. The effective Lagrangian (85) is of the Heisenberg–Euler type, which has been found for the case of the polarized vacuum of particles with arbitrary spins and magnetic moment. Here we took into account virtual arbitrary-spin particles, but not virtual photons. This is because at small energies of the external fields, the radiative corrections are small quantities. It is not difficult to find the asymptotic form of (83) for supercritical fields $eE/m^2 \to \infty$ and $eH/m^2 \to \infty$. It should be noted, however, that for strong electromagnetic fields, the anomalous magnetic moment of electrons depends on the external field [56, 57] and hence there is a similar dependence for arbitrary-spin particles. Therefore, to obtain the correct limit, it is necessary to take into account this dependence [39].

It is possible to obtain the imaginary part of the Lagrangian (78) from (83) using the residue theorem, taking into account the poles of expression (83) and passing above them [31].

6. DISCUSSION OF RESULTS

The theory of particles with arbitrary spins and magnetic moment based on Eq. (1) and the corresponding Lagrangian allows us to find density matrices (34), (35), (55), and (56), the pair-production probability (76), and the effective Lagrangian for electromagnetic fields (85), taking into account the polarization of the vacuum. It is convenient to use matrix-dyads (34), (35), (55), and

(56) for different electrodynamic calculations in the presence of particles with arbitrary spins. The exact formula for the intensity of pair production of arbitrary-spin particles coincides with the expression obtained in ref. 36 using the quasiclassical method of "imaginary time," which is valid only for $E, H \ll m^2/e$, i.e., for weak fields. Hence it follows that the analysis in ref. 36 is valid for arbitrary electromagnetic fields and that it is grounded in relativistic quantum field theory. In particular, there is a pair production by a purely magnetic field if $gs > 1$ [36], and in the presence of the magnetic field, the probability decreases for scalar particles and increases for higher spin particles. As all divergences and the renormalizability are contained in Re $\mathscr{L}$ of (83), but not in Im $\mathscr{L}$, the pair-production probability does not depend on the renormalization scheme. The vacuum polarization corrections for scalar and spinor (with $g = 2$) particles are reliable because their theories are renormalizable. The general formula (83) is of interest for the further development of the field theory of particles with higher spins (see discussion in refs. 36 and 37). Expression (83) is a reasonable result for arbitrary values of s and q because for the particular case of scalar and spinor particles, we arrive at known results. Thus, we have here a reasonable and noncontradictory description of the nonlinear effects that arise in this interaction.

REFERENCES

1. G. Felder, L. Kofman, and A. Linde (1999). *Phys. Rev. D* **60**, 103505.
2. I. L. Buchbinder and Sh. M. Shvartsman, (1993). *Int. J. Mod. Phys. A* **8**, 683.
3. F. Wilczek (1990). *Fractional Statistics and Anyon Superconductivity* (World Scientific, Singapore).
4. J. L. Cortes and M. S. Plyushchay (1996). *Int. J. Mod. Phys. A* **11**, 3331.
5. S. I. Kruglov (1997). *Mod. Phys. Lett. A* **12**, 1699.
6. S. I. Kruglov (1998). *Europhys. Lett.* **44**, 361.
7. S. I. Kruglov and M. N. Sergeenko (1997). *Mod. Phys. Lett. A* **12**, 2475.
8. H. J. Bhabha (1945). *Rev. Mod. Phys.* **17**, 200.
9. M. Fierz and W. Pauli (1939). *Proc. R. Soc. Lond. A* **173**, 211.
10. W. Rarita and J. Schwinger (1941). *Phys. Rev.* **60**, 61.
11. V. Bargman and E. Wigner (1948). *Proc. Am. Acad. Sci.* **34**, 211.
12. M. A. K. Khalil (1977). *Progr. Theor. Phys.* **58**, 1538.
13. W. J. Hurley (1971). *Phys. Rev. D* **4**, 3605.
14. W. J. Hurley (1974). *Phys. Rev. D,* **10**, 1185.
15. W. J. Hurley (1972). *Phys. Rev. Lett.* **29**, 1475.
16. G. Velo and D. Zwanziger (1969). *Phys. Rev.* **186**, 1337.
17. K. Johnson and E. C. G. Sudarshan (1961). *Ann. Phys.* **13**, 126.
18. W. K. Tung (1966). *Phys. Rev. Lett.* **16**, 763.
19. W. K. Tung (1967). *Phys. Rev.* **156**, 1385.
20. S. J. Chang (1966). *Phys. Rev. Lett.* **17**, 1024.
21. R. P. Feynman and M. Gell-Mann (1958). *Phys. Rev.* **109**, 193.
22. D. V. Ahluwalia and D. J. Ernst (1992). *Phys. Lett. B* **287**, 18.
23. D. V. Ahluwalia and D. J. Ernst (1992). *Phys. Rev. C* **45**, 3010.

24. D. V. Ahluwalia and D. J. Ernst (1993). *Int. J. Mod. Phys. E.* **2**, 397.
25. D. V. Ahluwalia, T. Goldman, and M. B. Jonson (1994). *Mod. Phys. Lett. A* **9**, 439.
26. D. V. Ahluwalia (1996). *Int. J. Mod. Phys A* **11**, 1855.
27. V. V. Dvoeglasov (1998). *Int. J. Theor. Phys.* **37**, 1915.
28. F. Sauter (1931). *Z. Physik* **69**, 742.
29. F. Sauter (1931). *Z. Physik* **73**, 547.
30. J. M. Jauch and F. Rohrlich (1955). *The Theory of Photons and Electrons* (Addison-Wesley, Cambridge, Massachusetts).
31. J. Schwinger (1951). *Phys. Rev.* **82**, 664.
32. Ya. B. Zel'dovich, L. P. Pitaevskii, V. S. Popov, and A. A. Starobinskii (1972). *Sov. Phys.-Usp.* **14**, 811 [*Usp. Fiz. Nauk* **105**, 781 (1971)].
33. A. I. Nikishov (1970). *Sov. Phys.-JETP* **30**, 660 [*Zh. Eksp. Teor. Fiz.* **57**, 1210 (1969)].
34. A. I. Nikishov (1970). *Nucl. Phys. B* **21**, 346.
35. A. Nikishov (1979). *Trudy FIAN* **111**, 152.
36. M. S. Marinov and V. S. Popov (1972). *Sov. J. Nucl. Phys.* **15**, 702 [*Yad. Fiz.* **15**, 1271 (1972)].
37. V. S. Vanyashin and M. V. Terent'ev (1965). *Sov. Phys.-JETP* **21**, 375 [*Zh. Eksp. Teor. Fiz.* **48**, 565 (1965)].
38. W. Dittrich (1978). *J. Phys. A Math. Gen.* **11**, 1191.
39. P. M. Lavrov (1987). *Izv. Vuz. Fiz.* **3**, 37.
40. A. Proca (1936). *Compt. Rend.* **202**, 1420.
41. E. Petiau (1936). Thesis, Paris.
42. R. J. Duffin (1938). *Phys. Rev.* **54**, 1114.
43. H. Kemmer (1939). *Proc. R. Soc.* **173,** 91.
44. I. M. Gel'fand, R. A. Minlos, and Z. Ya. Shapiro, (1963). *Representations of the Rotation and Lorentz Groups and Their Applications* (Pergamon Press, New York).
45. M. A. Naimark (1964). *Linear Representations of the Lorentz Group* (Pergamon Press, New York).
46. Harish-Chandra (1947). *Phys. Rev.* **71**, 793.
47. W. J. Hurley and E. C. G. Sudarshan (1974). *Ann. Phys.* (N. Y.) **85**, 546.
48. F. I. Fedorov (1958). *Zh. Eksp. Teor. Fiz.* **35**, 495.
49. F. I. Federov (1979). *The Lorentz Group* (Nauka, Moscow).
50. N. Cabibbo and E. Ferrari (1962). *Nuovo Cim.* **23**, 1147.
51. A. A. Grib, S. G. Mamayev, and V. M. Mostepanenko (1980). *Quantum Effects in Intensive External Fields* (Atomizdat, Moscow).
52. S. I. Kruglov (1995). *J. Phys. G. Nucl. Phys.* **21**, 1643.
53. S. I. Kruglov (1996). *J. Phys. G. Nucl. Phys.* **22**, 461.
54. W. Heisenberg and H. Euler (1936). *Z. Physik* **98**, 714.
55. V. Weisskopf (1936). *Kgl. Danske Videnskab Selskab, Mat.-Fys. Medd.* **14**, 6.
56. I. M. Ternov and V. G. Bagrov (1968). *Zh. Eksp. Teor. Fiz.* **55**, 2273.
57. V. N. Bayer, B. M. Katkov, and V. M. Strachovenko (1976). *Yad. Fiz.* **24**, 379.